

# Group theoretical examination of the relativistic wave equations on curved spaces.

## III. Real reducible spaces

S.A.Pol'shin

### Abstract

The group theoretical approach to the relativistic wave equations on the real reducible spaces for spin 0, 1/2 and 1 massless particles is considered. The invariant wave equations which determine the appropriate irreducible representations are constructed. The coincidence of these equations with the general-covariant Klein-Gordon, Weyl and Maxwell equations on the corresponding spaces is shown. The explicit solutions of these equations possessing a simplicity and physical transparency are obtained in the form of so-called "plane waves" without using the method of separation of variables. The invariance properties of these "plane waves" for the spinless particles under the group  $SO(3,1)$  were used for the construction of the invariant spin 0,1/2 and 1 two-point functions on the  $H^3$ . Secondly quantized spin 0,1/2 and 1 fields on the  $\mathbb{R}^1 \otimes H^3$  are constructed; their propagators which are their (anti)commutators in different points, are expressed in terms of the mentioned two-point functions. From here the  $\mathbb{R}^1 \otimes SO(3,1)$ -invariance follows.

## 1 Introduction

In the present paper we consider the relativistic wave equations for the massless particles in real reducible spaces following the program outlined in [1]. The Einstein space  $\mathbb{R}^1 \otimes S^3$  is the most important of them; the wave equations on this space were considered in a number of papers by M.Carmeli with co-authors using some group-theoretical methods (see [2]). We will devote the main part of our paper to the space  $\mathbb{R}^1 \otimes H^3$  which differs from the space  $\mathbb{R}^1 \otimes S^3$  only by the sign of curvature. Our choice based on two important advantages of the space  $\mathbb{R}^1 \otimes H^3$  over the space  $\mathbb{R}^1 \otimes S^3$ . The first advantage consists in that the symmetry group of the space  $\mathbb{R}^1 \otimes H^3$  has infinite-dimensional unitary irreducible representations; the second one consists in that the solutions of the group-theoretical wave equations on the space  $\mathbb{R}^1 \otimes H^3$  possess the remarkable invariance properties under the symmetry group of the space, which allows us to construct invariant propagators with the help of the mentioned solutions.

The present paper is constructed as follows. In Sect.2 the space  $H^3$ , its symmetry group and its irreducible representations are considered. Sect.3 is devoted to the massless spin zero

particles on the  $\mathbb{R}^1 \otimes H^3$ . It is shown that the corresponding invariant wave operator coincides with the covariant d'Alembertian; the analogous situation takes place in the de Sitter space [3] too. Sect.4 treats the wave equations for the massless spin 1/2 and 1 fields. In Sect.5 we consider the exact solutions of these equations in the form of so-called "plane waves". In Sect.6 is shown that the obtained group-theoretical wave equations for the spin 1/2 and 1 particles coincides with the general-covariant Weyl and Maxwell equations on the  $\mathbb{R}^1 \otimes H^3$ .

In Sect.7 we apply the obtained exact solutions of wave equations to the construction of the secondly-quantized spin 0, 1/2 and 1 fields on the  $\mathbb{R}^1 \otimes H^3$  and their invariant propagators. The obtaining of invariant propagators is an important problem of the quantum field theory on the curved spaces. Earlier the propagators of various spin fields on the de Sitter and Anti-de Sitter spaces (see [4, 5, 6] and references therein) and spin 0 and 1/2 fields on the  $S^n$  [7] were considered. In the recent paper [8] the propagators for massive spin 0 and 1/2 fields over the  $\mathbb{R}^1 \otimes S^n$  space are constructed. In the above referred papers the propagators are constructed as invariant functions obeys the corresponding wave equations. In this case its connection with the (anti)commutators of secondly quantized fields is unclear. In the Sect.7 we start from the invariance properties of the obtained "plane waves", which properties based on the connection of these "plane waves" with the generalized coherent states for the group  $SO(3, 1)$ . Using these properties we construct the invariant spin 0, 1/2 and 1 two-point functions on the  $H^3$ . The propagators of these fields being their (anti)commutators in the different points, are expressed through the mentioned two-point functions and possess the invariance under the symmetry group of our space.

In Sect.8 we consider the wave equations on the other reducible spaces: Einstein space and one more space with the symmetry group  $T_1^{\mathbb{R}} \otimes SO(3)$ .

Indices  $\alpha, \beta, \dots$  correspond to the embracing space for  $H^3$  and run values 1 up to 4; indices  $\mu, \nu, \dots$  correspond to the space  $\mathbb{R}^1 \otimes H^3$  and run values 0 up to 3. Indices  $i, k, \dots = 1, 2, 3$ .

## 2 $H^3$ space and its symmetry group

The  $H^3$  space is the hypersphere in the fictitious four-dimensional pseudo-euclidean space with the metric  $\eta_{\alpha\beta} = \text{diag} (+1, +1, +1, -1)$ ; this hypersphere is determined by the equation

$$\eta_{\alpha\beta} x^\alpha x^\beta = -R^2. \quad (2.1)$$

As the interval looks like  $ds^2 = dx^\alpha dx_\alpha$ , then, by expressing  $x^4$  from (2.1):

$$x^4 = \varkappa \equiv \sqrt{1 + \mathbf{x}^2/R^2},$$

we obtain

$$\begin{aligned} ds^2 &= g_{ij} dx^i dx^j \\ g_{ij} &= \delta_{ij} - \frac{x^i x^j}{R^2 \varkappa^2} \\ g^{ij} &= \delta^{ij} + \frac{x^i x^j}{R^2}. \end{aligned} \tag{2.2}$$

By the choice of  $x^4$  sign, we have fixed one of the halves of the space, which is covered by our coordinates. The symmetry group of the space is, obviously,  $SO(3, 1)$  with generators of rotations  $\mathbf{X}$  and translations  $\mathbf{P}$  and commutation relations

$$[P_i, P_j] = -R^{-2} \varepsilon_{ijk} X_k \tag{2.3}$$

$$[X_i, X_j] = \varepsilon_{ijk} X_k \tag{2.4}$$

$$[P_i, X_j] = \varepsilon_{ijk} P_k. \tag{2.5}$$

With  $R \rightarrow \infty$  these commutation relations turn into commutation relations of the three-dimensional Euclidean group.

Our group has two second-order Casimir operators:

$$\begin{aligned} C_2^{(1)} &= P_i P_i - R^{-2} X_i X_i \\ C_2^{(2)} &= P_i X_i. \end{aligned}$$

It is easy to show that the generators

$$\Pi_i^\pm = P_i \pm iR^{-1} X_i \tag{2.6}$$

compose two  $SO(3)$ -like subgroups commuting with each other:

$$\begin{aligned} [\Pi_i^\pm, \Pi_j^\pm] &= \pm \frac{2i}{R} \varepsilon_{ijk} \Pi_k^\pm; \\ [\Pi_i^+, \Pi_j^-] &= 0. \end{aligned}$$

Since the presence of the imaginary unit in (2.6) the isomorphism of the  $SO(3, 1)$  and  $SO(3) \otimes SO(3)$  takes place only at a level of the Lie algebras. The Casimir operators of these subgroups

$$-R^{-2} C_2^\pm = \frac{1}{4} \Pi_i^\pm \Pi_i^\pm = \frac{1}{4} C_2^{(1)} \pm \frac{1}{2} i R^{-1} C_2^{(2)} \tag{2.7}$$

are the Casimir operators of the all group. The irreducible representations of the  $SO(3, 1)$  group are well-known [9]; they are determined by two (complex, in general) numbers  $j_+$  and  $j_-$  such as

$$j_+ - j_- = \pm s, \quad s = 0, 1/2, 1, \dots \tag{2.8}$$

From the Weinberg theorem [10] it follows that such a representation should describe the spin  $s$  particles. If  $j_+ + j_-$  also is an integer or half-integer then the representation is finite-dimensional and non-unitary; in the opposite case the representation is infinite-dimensional and unitary. In the representation  $\mathbb{L}^{(j_+, j_-)}$  the eigenvalues of the Casimir operators are

$$C_2^\pm = -j_\pm(j_\pm + 1). \quad (2.9)$$

As the operators  $C_2^{(1)}$  and  $C_2^{(2)}$  are Hermitian then those eigenvalues should be real; from here follows  $C_2^+ = \overline{C_2^-}$ ; combining with (2.9) one can obtain

$$(\overline{j}_+ - j_-)(\overline{j}_+ + j_- + 1) = 0.$$

If the first bracket is equal to zero then we obtain the additional series of representations which is not interest for us. If the second bracket is equal to zero then we came to the principal series of representations and from (2.8) follows

$$\operatorname{Re} j_+ = \frac{\pm s - 1}{2} \quad , \quad \operatorname{Re} j_- = \frac{\mp s - 1}{2}. \quad (2.10)$$

The metric on the  $\mathbb{R}^1 \otimes H^3$  is determined by the formulas (2.2) and  $g_{00} = -1$ ,  $g_{0i} = 0$ .

### 3 Spinless particles and Klein-Gordon equation

As for the metric (2.2) the Christoffel symbols, not equal to zero, are

$$\Gamma_{ij}^k = R^{-2} g_{ij} x^k,$$

then it is easy to obtain the Killing vectors, which give us the generators of scalar representation:

$$P_i^{(l)} = \varkappa \partial_i \quad , \quad X_i^{(l)} = \varepsilon_{ijk} x^k \partial_j. \quad (3.1)$$

Therefore, the generators of scalar representation of both  $SO(3)$ -subgroups looks like

$$\Pi_i^{+(l)} = e_{(i)}^k \partial_k \quad , \quad \Pi_i^{-(l)} = \overline{e}_{(i)}^k \partial_k, \quad (3.2)$$

where we introduced the complex orthonormal dreibein

$$e_{(i)}^k = \varkappa \delta_i^k + i R^{-1} \varepsilon_{ikl} x^l \quad , \quad e_{(i)}^l e_{(k)l} = \delta_{ik}. \quad (3.3)$$

Combining (3.2) and (2.7) yields

$$\begin{aligned} -4R^{-2} C_2^{+(l)} &= g^{ik} \partial_i \partial_k + e^{(i)l}{}_{,k} e_{(i)}^k \partial_l \\ -4R^{-2} C_2^{-(l)} &= g^{ik} \partial_i \partial_k + \overline{e}^{(i)l}{}_{,k} \overline{e}_{(i)}^k \partial_l. \end{aligned}$$

To evaluate of the second term in the right hand side let us write

$$e^{(i)l}_{,k} e^k_{(i)} = (e^{(i)l}_{,k} - \Gamma^l_{km} e^{(i)m}) e^k_{(i)} = G^i_{ni} e^{(n)l} - \Gamma^l_{km} g^{km}, \quad (3.4)$$

where the Ricci rotational coefficients  $G_{ikl} = e_{(i)m;n} e^m_{(k)} e^n_{(l)}$  are introduced. In our case they equal to

$$G_{ikl} = iR^{-1} \varepsilon_{ikl}. \quad (3.5)$$

Using (3.5) and the known general formula for the rolling up of the Christoffel symbols

$$\Gamma^l_{km} g^{km} = -g^{-1/2} \partial_k (g^{1/2} g^{lk}), \quad (3.6)$$

where  $g = \det g_{ik}$ , we obtain the Casimir operators of the scalar representation:

$$-4R^{-2} C_2^{+(l)} = -4R^2 C_2^{-(l)} = g^{-1/2} \partial_i \{g^{1/2} g^{ik} \partial_k\} \equiv \Delta. \quad (3.7)$$

This result does not depend(hold) on the manifest form of vierbeins, but only on correctness of (3.5). We choose the representation with the weights  $j_+ = j_- = \frac{1}{2}(i\omega R - 1)$ , where  $\omega \in \mathbb{R}$ . Then by calculating of the eigenvalues of the Casimir operators we obtain the Laplace equation:

$$(\Delta + \omega^2 + R^{-2})\psi = 0$$

which describes the field of a mass  $\omega$  in the  $H^3$  space. In the space  $\mathbb{R}^1 \otimes H^3$  the quantity  $\omega$  play a part of the frequency:  $\psi \sim e^{i\omega t}$  and we obtain the Klein-Gordon equation

$$(\square + R^{-2})\psi = 0 \quad , \quad \square \equiv (-\det g_{\mu\nu})^{-1/2} \partial_\mu ((-\det g_{\mu\nu})^{1/2} g^{\mu\nu} \partial_\nu) = -\frac{\partial^2}{\partial t^2} + \Delta. \quad (3.8)$$

for the conformally-coupled scalar field over  $\mathbb{R}^1 \otimes H^3$ .

## 4 Spin-1/2 and 1 particles; Weyl and Maxwell equations

Let us take a certain matrix representation of the rotation group with generators  $\mathbf{X}^{(s)}$  which corresponds to the spin  $s$ . Then the generators

$$\mathbf{P}^{(s)} = \frac{i}{R} \mathbf{X}^{(s)}$$

obey the commutation relations (2.3),(2.5); the corresponding representation of the  $SO(3,1)$  is  $\mathbf{L}^{(s,0)}$ . For the spin 1/2 representation

$$\begin{aligned} \mathbf{X}^{(s)} &= -\frac{i\boldsymbol{\sigma}}{2} \\ X^{(s)i} X^{(s)k} &= -\frac{1}{4} \delta^{ik} + \frac{1}{2} \varepsilon^{ikl} X^{(s)l}. \end{aligned} \quad (4.1)$$

For the spin 1 representation

$$\begin{aligned} (X_i^{(s)})_{kl} &= -\varepsilon_{ikl} \\ (X_i^{(s)} X_k^{(s)})_{mn} &= -\delta_{ik} \delta_{mn} + \delta_{in} \delta_{km}. \end{aligned} \quad (4.2)$$

Now, let us consider the representation with the generators  $\mathbf{P}^{(l)} + \mathbf{P}^{(s)}$  and  $\mathbf{X}^{(l)} + \mathbf{X}^{(s)}$ . The corresponding Casimir operators are

$$\begin{aligned} -4R^{-2}C_2^+ &= \Delta + \frac{4s}{R}A + \frac{4}{R^2}s(s+1) \\ -4R^{-2}C_2^- &= \Delta, \end{aligned} \quad (4.3)$$

where we use (3.7) and designate the spinor wave operator on  $H^3$  as

$$A = is^{-1}X^{(s)i}e_{(i)}^k\partial_k.$$

Let us square the above expression:

$$-s^2A^2 = X_i^{(s)}X_k^{(s)}e_{(i)}^me_{(k)}^n(\partial_m\partial_n - \Gamma_{mn}^l\partial_l) + \frac{s}{R}A,$$

where we expressed the derivatives of dreibein through  $G_{ikl}$  and  $\Gamma_{mn}^l$  by analogy with (3.4). Using (4.2) and (3.6) we obtain for the case  $s = 1/2$ :

$$A^2 = \Delta - \frac{2A}{R}. \quad (4.4)$$

From the other hand, using (2.4) and (3.5) gives

$$-s^2A^2 = X_k^{(s)}X_i^{(s)}e_{(i)}^m\partial_m(e_{(k)}^n\partial_n) + \frac{2s}{R}A.$$

Substituting (4.1) in the above expression, we obtain for the spin 1 in the component-wise form

$$-(A^2\psi)_i = -\Delta\psi_i + e_{(i)}^m\partial_m(e_{(k)}^n\partial_n\psi^k) + \frac{2}{R}(A\psi)_i.$$

As both  $\Delta$  and  $A$  should have fixed eigenvalues in the irreducible representations, then (4.4) holds as before and the gauge condition

$$e_{(i)}^k\partial_k\psi^i = 0. \quad (4.5)$$

should be satisfied. Using (2.8),(2.9),(2.10),(4.3) and (4.4) we obtain that our representations has the weights

$$j_+ = -\frac{i\omega R + s + 1}{2}, \quad j_- = \frac{-i\omega R + s - 1}{2},$$

where  $\omega \in \mathbb{R}$  is a frequency. The eigenvalues of the wave operators are

$$A\psi = \left(i\omega - \frac{s+1}{R}\right)\psi$$

$$\left(\Delta - \frac{(i\omega R - s)^2 - 1}{R^2}\right)\psi = 0.$$

Proceeding to the space  $\mathbb{R}^1 \otimes H^3$  we obtain that both for spin 1/2 particles and for spin 1 particles the equation

$$(\partial_t - is^{-1}X^{(s)i}e_{(i)}^k\partial_k - \frac{s+1}{R})\psi = 0, \quad (4.6)$$

is correct as well as the gauge condition (4.5) for spin 1 particles is. The obtained equations are the Weyl equations for the neutrino and the Maxwell equations for the photon. If we start from the matrix representations  $\mathbf{L}^{(0,s)}$  then we came to the equation

$$(\partial_t + is^{-1}X^{(s)i}e_{(i)}^k\partial_k + \frac{s+1}{R})\psi = 0 \quad (4.7)$$

which corresponds to the antineutrino and photon.

## 5 Explicit solutions of the wave equations

Let  $\mathbf{k}$  is a unit 3-vector:  $\mathbf{k}^2 = 1$ ; let us consider the  $2s + 1$ -component quantities  $u_s^\pm(\mathbf{k})$  which obey the condition

$$(is^{-1}\mathbf{k}\mathbf{X}^{(s)} \pm 1)u_s^\pm(\mathbf{k}) = 0.$$

In the case  $s = 0$   $u(\mathbf{k}) \equiv 1$ . In the case of spin 1 we also impose the one more condition, thus

$$\mathbf{u}^2 = \mathbf{k}\mathbf{u} = 0. \quad (5.1)$$

The explicit form of the  $u_s^\pm(\mathbf{k})$  see in Appendix. Now, let us construct the functions

$$\varphi_{\mathbf{k}\omega}^{(s)\pm}(\mathbf{x}) = \left(\varkappa - \frac{\mathbf{k}\mathbf{x}}{R}\right)^{i\omega R - s - 1} u_s^\pm(\mathbf{k}).$$

One can omit the symbol  $\pm$  at the functions  $\varphi_{\mathbf{k}\omega}^{(0)\pm}(\mathbf{x})$ . It is easy to show that the functions

$$\psi(\mathbf{x}, t) = e^{i\omega t} \varphi_{\mathbf{k}\omega}^{(s)+}(\mathbf{x}) \quad (5.2)$$

obey Eq. (3.8) for  $s = 0$ , (4.6) for  $s = 1/2$  and (4.5),(4.6) for  $s = 1$ . In two last cases it is necessary to use the equality

$$(\varepsilon_{ijl}X_i^{(s)}k^jx^l + s\mathbf{k}\mathbf{x} \pm i\mathbf{X}^{(s)}\mathbf{x})u_s^\pm(\mathbf{k}) = 0,$$

that we prove easy taking into account the explicit form of the generators and the condition (5.1). The functions

$$\psi(\mathbf{x}, t) = e^{i\omega t} \varphi_{\mathbf{k}\omega}^{(s)-}(\mathbf{x}) \quad (5.3)$$

obey the Eq. (4.7)

$$(\partial_t + is^{-1} X^{(s)i} \bar{e}_{(i)}^k \partial_k - \frac{s+1}{R})\psi = 0 \quad (5.4)$$

for the antiparticles in the complex-conjugate dreibein. To prove that in the spin 1 case the solution (5.2) really describes the "plane" electromagnetic wave let us divide  $\psi$  onto the electrical and magnetic fields:

$$\psi = \mathbf{E} + i\mathbf{H}.$$

As  $\psi^2 = \mathbf{k}\psi = 0$  then  $\mathbf{E}^2 = \mathbf{H}^2$  and thus  $\mathbf{E}, \mathbf{H}$  and  $\mathbf{k}$  are orthogonal to each other, i.e. our solution really describes the "plane" electromagnetic wave. Functions  $\varphi_{\mathbf{k}\omega}^{(s)\pm}(\mathbf{x})$  are analogous to the "plane waves" on the de Sitter space [11, 3]. With  $R \rightarrow \infty$  functions (5.2) pass onto the usual plane waves over the Minkowski space:

$$\lim_{R \rightarrow \infty} e^{i\omega t} \varphi_{\mathbf{k}\omega}^{(s)\pm}(\mathbf{x}) = \exp(ip_\mu x^\mu) u_s^\pm(\mathbf{p}),$$

where  $p^\mu = (\omega, \omega\mathbf{k})$ . The solutions we have obtained are much simpler than the solutions of the spin 0 and 1/2 wave equations in the Einstein space obtained for the spin 1/2 by the method of the separation of variables in [12] and for the spin 0, 1/2 and 1 using some group theoretical methods in [2].

## 6 Comparison with the covariant approach

Let us complete the dreibein (3.3) to the vierbein by

$$e_{(i)}^0 = e_{(0)}^i = 0 \quad , \quad e_{(0)}^0 = 1, \quad (6.1)$$

then  $e_{(\rho)}^\mu e^{(\rho)\nu} = g^{\mu\nu}$ . Substituting this in the massless general-covariant Dirac equation one can obtain

$$\begin{aligned} i\gamma^\mu e_{(\mu)}^\nu \mathcal{D}_\nu \psi &= 0 \\ \mathcal{D}_\nu &= \partial_\nu - \frac{1}{4} G_{\rho\sigma\mu} \gamma^\rho \gamma^\sigma e_\nu^{(\mu)}. \end{aligned} \quad (6.2)$$

The possibility of using the complex vierbein (3.3), (6.1) in the Weyl and Dirac equations is proved by the fact that the coincidence of the vector and spinor  $SO(3)$  transformations

$$L_{ik} = \delta_{ik} \cos \vartheta + \varepsilon_{ikj} l^j \sin \vartheta + 2l^i l^k \sin^2 \vartheta / 2 \Leftrightarrow U(g) = \exp(-i\vartheta \boldsymbol{\sigma} \mathbf{l} / 2), \quad (6.3)$$

where  $\mathbf{l}^2 = 1$  so that

$$L_{ik}(g) U(g) \sigma_k U^{-1}(g) = \sigma_i \quad , \quad g \in SO(3),$$



may be formally continued to the area of complex rotation parameters. Since the complex conjugation is nowhere used in the (6.2) and (6.3), then Eq. (6.2) is invariant under the complex rotations of the vierbein. However, the vierbein (3.3),(6.1) may be transformed into the real vierbein

$$e_{(i)}^k = \delta_{ik} + \frac{x^i x^k}{R^2(\varkappa + 1)} \quad , \quad e_{(i)}^0 = e_{(0)}^i = 0 \quad , \quad e_{(0)}^0 = 1$$

with the help of rotations around the unit vector  $\mathbf{l} = \mathbf{x}/r$  on the imaginary angle  $\vartheta$ ,  $\cos \vartheta = \varkappa$ .

Splitting (6.2)<sup>1</sup>, we obtain two equations which coincides with (4.6) and (4.7) for the spin 1/2. By this the general-covariance of the mentioned equations is proven also. Let us use this general-covariance for the transformation of the functions (5.3) from the vierbein  $\bar{e}_{(\mu)}^\nu$  to the vierbein  $e_{(\mu)}^\nu$ . The corresponding three-dimensional transformation is the rotation of the form (6.3) around  $\mathbf{l} = \mathbf{x}/r$  on the angle  $\vartheta$ ,  $\sin \vartheta/2 = ir/R$ . Corresponding spinor transformation is

$$V(\mathbf{x}) = \varkappa - \frac{\boldsymbol{\sigma} \mathbf{x}}{R}.$$

Then the functions

$$\psi(\mathbf{x}, t) = e^{i\omega t} V(\mathbf{x}) \varphi_{\mathbf{k}\omega}^{(1/2)-}(\mathbf{x}) \quad (6.4)$$

obey Eq. (4.7) for the spin 1/2.

Now, let us consider the spin 1 particles. Let us introduce the vierbein tensor of the electromagnetic field

$$\Phi_{\mu\nu} = e_{(\mu)}^\rho e_{(\nu)}^\sigma F_{\rho\sigma},$$

where  $F_{\rho\sigma}$  is the usual tensor of the electromagnetic field. By expressing the covariant derivatives of  $F_{\mu\nu}$  in terms of vierbein derivatives of  $\Phi_{\rho\sigma}$  we obtain the Maxwell equations

$$\begin{aligned} e_{(\nu)}^\sigma \partial_\sigma \Phi^{\mu\nu} - G^{\mu\nu\rho} \Phi_{\nu\rho} - \Phi^{\sigma\mu} G_{\sigma\varkappa}^\varkappa &= 0 \\ \varepsilon^{\rho\mu\nu\varkappa} (e_{(\varkappa)}^\sigma \partial_\sigma \Phi_{\mu\nu} + 2\Phi_{\nu}^\sigma G_{\sigma\mu\varkappa}) &= 0 \end{aligned} \quad (6.5)$$

Substituting (3.5) into (6.5) and introducing the complex vector of the electromagnetic field

$$\psi_i = \Phi_{i0} + \frac{i}{2} \varepsilon_{ikl} \Phi_{kl},$$

we obtain that (6.5) coincides with (4.5) and (4.6) for the spin 1 case.

## 7 Propagators and two-point functions

The functions  $\varphi_{\mathbf{k}\omega}^{(0)}(\mathbf{x})$  are the generalized coherent states for the group  $SO(3, 1)$  [13]; from here its nontrivial transformation properties follow. Let us juxtapose to each element  $g \in SO(3, 1)$

---

<sup>1</sup> Also it is convenient to pass to the signature  $(+1, -1, -1, -1)$  temporarily, in order to have the possibility of using the usual  $\gamma$ -matrices in the standard representation.

the matrix  $L^\alpha_\beta(g)$  of the orthogonal transformations of the space  $H^3$ . Then we can define the action of the group  $SO(3, 1)$  over the space  $H^3$  and over the null 4-vectors  $n^\alpha$ ,  $n^\alpha n_\alpha = 0$ :

$$x^\alpha \mapsto x^\alpha_g = L^\alpha_\beta(g)x^\beta, \quad n^\alpha \mapsto n^\alpha_g = L^\alpha_\beta(g)n^\beta. \quad (7.1)$$

Then the unit 3-vector  $\mathbf{k}$  can be expressed through  $n^\alpha$ :  $\mathbf{k} = \mathbf{n}/n^4$ , which gives us the action of the group  $SO(3, 1)$  on the unit 3-vectors:  $\mathbf{k} \mapsto \mathbf{k}_g$ . Then it is easy to show that the functions  $\varphi^{(0)}_{\mathbf{k}\omega}(\mathbf{x})$  transform under the transformations (7.1) as follows:

$$\varphi^{(0)}_{\mathbf{k}\omega}(\mathbf{x}_g) = (L_4^4(g) + L_4^i(g)k^i)^{-i\omega R-1} \varphi^{(0)}_{\mathbf{k}'\omega}(\mathbf{x}), \quad (7.2)$$

where  $\mathbf{k}' = \mathbf{k}_{g^{-1}}$  and  $\lambda \in \mathbb{C}$ . The correctness of the above expression under the  $g \in SO(3)$  is obvious. It is necessary to prove this only for the transformations  $g = g_\xi$ , which is given by the matrices

$$L_4^4(g) = (1 + \xi^2)^{1/2}, \quad L_4^i(g) = L^{4i}(g) = \xi^i, \quad L_{ik} = \delta_{ik} + \frac{\xi^i \xi^k}{1 + \sqrt{1 + \xi^2}}. \quad (7.3)$$

By calculating the Jacobian of the transformation from the  $\mathbf{k}$  to the  $\mathbf{k}'$  it is easy to show that the two-point function

$$\mathcal{W}^{(0)}(\mathbf{x}, \mathbf{y}; \omega) = \int_{S^2} d^2k \varphi^{(0)}_{\mathbf{k}\omega}(\mathbf{x}) \varphi^{(0)}_{\mathbf{k}, -\omega}(\mathbf{y}),$$

where  $\mathbf{x}$  and  $\mathbf{y}$  are the arbitrary points of the upper half of the space  $H^3$  and  $d^2k = dk^1 dk^2 / k^3$  is an invariant measure, possesses the  $SO(3, 1)$ -invariance:

$$\mathcal{W}^{(0)}(\mathbf{x}_g, \mathbf{y}_g; \omega) = \mathcal{W}^{(0)}(\mathbf{x}, \mathbf{y}; \omega). \quad (7.4)$$

In fact,  $\mathcal{W}^{(0)}(\mathbf{x}_g, \mathbf{y}_g; \omega)$  is the scalar product of the coherent states corresponding to the points  $\mathbf{x}$  and  $\mathbf{y}$ . Then the equality

$$\mathcal{W}^{(0)}(\mathbf{x}, \mathbf{y}; \omega) = F\left(-\frac{i\omega R - 1}{2}, \frac{i\omega R + 1}{2}, \frac{3}{2}; 1 - \left(\frac{x_\alpha y^\alpha}{R}\right)^2\right).$$

is correct [13].

The above considerations may be generalized to the cases of spin 1/2 and 1 too. Let us construct the  $(2s + 1) \times (2s + 1)$ -matrices

$$\Sigma(\mathbf{k}; s) = u_s^+(\mathbf{k}) \otimes u_s^{+\dagger}(\mathbf{k}).$$

We can show (see Appendix) the validity of

$$U_s(g) \Sigma(\mathbf{k}; s) U_s^\dagger(g) = \left(\frac{n_g^4}{n^4}\right)^{2s} \Sigma(\mathbf{k}_g; s), \quad (7.5)$$

where  $U_s(g)$  is the matrix of the representation  $\mathbf{L}^{(s,0)}$  which corresponds to the transformation  $g \in SO(3,1)$ . Then, by the way completely analogous to that for the spin zero particles and using (7.5) we can obtain that the two-point functions

$$\mathcal{W}^{(s)}(\mathbf{x}, \mathbf{y}; \omega) = \int_{S^2} d^2k \varphi_{\mathbf{k}\omega}^{(s)+}(\mathbf{x}) \otimes \left( \varphi_{\mathbf{k}\omega}^{(s)+}(\mathbf{y}) \right)^\dagger$$

possess the  $SO(3,1)$ -invariance:

$$\mathcal{W}^{(s)}(\mathbf{x}_g, \mathbf{y}_g; \omega) = U_s(g) \mathcal{W}^{(s)}(\mathbf{x}, \mathbf{y}; \omega) U_s^\dagger(g).$$

Now, to clarify of the physical meaning of the constructed two-point functions let us construct now the secondly quantized massless spin 0, 1/2 and 1 fields:

$$\Psi^{(s)}(x) = \int_0^\infty \frac{\omega d\omega}{(2\pi)^{3/2}} \int_{S^2} d^2k \left( e^{i\omega x^0} \varphi_{\mathbf{k}\omega}^{(s)+}(\mathbf{x}) a(\mathbf{p}; s) + e^{-i\omega x^0} \varphi_{\mathbf{k}, -\omega}^{(s)+}(\mathbf{x}) a^\dagger(\mathbf{p}; s) \right),$$

where the bosonic and fermionic creation-destruction operators

$$[a_A(\mathbf{p}; s), a_B^\dagger(\mathbf{p}'; s)]_\pm = p^0 \delta^3(\mathbf{p} - \mathbf{p}') \delta_{AB}$$

are introduced and  $A, B = 1, \dots, 2s + 1$ . The choice of a measure is correct because

$$\omega d\omega d^2k = \frac{d^3\mathbf{p}}{p^0}.$$

Then the corresponding propagators

$$\begin{aligned} \Delta_{AB}^{(s)}(x, y) &\equiv [\Psi_A^{(s)}(x), \Psi_B^{(s)\dagger}(y)]_\pm = \\ &= \int_0^\infty \frac{\omega d\omega}{(2\pi)^3} \left( e^{i\omega(x^0 - y^0)} \mathcal{W}_{AB}^{(s)}(\mathbf{x}, \mathbf{y}; \omega) - e^{-i\omega(x^0 - y^0)} \mathcal{W}_{AB}^{(s)}(\mathbf{y}, \mathbf{x}; \omega) \right) \end{aligned}$$

possess the invariance under the time translations and spatial  $SO(3,1)$  transformations. The equality

$$\int \frac{d^3\mathbf{x}}{\varkappa} \varphi_{\mathbf{k}\omega}^{(0)}(\mathbf{x}) \varphi_{\mathbf{k}', -\omega'}^{(0)}(\mathbf{x}) = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}'),$$

is correct [13], where  $\mathbf{k}, \mathbf{k}', \omega, \omega'$  are arbitrary. Then, it is easy to show that the propagator for spin zero we have constructed possess the reproductivity property:

$$\int \frac{d^3\mathbf{x}}{\varkappa} \Delta^{(0)}(y, x) \Delta^{(0)}(x, z) = \Delta^{(0)}(y, z).$$

## 8 Other reducible spaces

The passage from the space  $\mathbb{R}^1 \otimes H^3$  to the Einstein space  $\mathbb{R}^1 \otimes S^3$  may be performed by the change of sign of  $\eta_{44}$  and by the replacement  $R \rightarrow iR$ . The symmetry group of  $S^3$  is  $SO(4) = SO(3) \otimes SO(3)$ ; the dreibein (3.3) becomes real, and therefore the operators  $\Pi_i^\pm$  becomes the generators of two subgroups of true translations of the space  $S^3$ . The invariant wave operators remains unchanged to within the replacement  $R \rightarrow iR$ , but their eigenvalues no longer may be obtained by the group-theoretical way. To within the same replacement, the explicit solutions of the wave equations and the general-covariant equations remain unchanged, but many results of Sect.6 don't take a place since the vierbein (3.3) becomes real. The results of Sect.7 don't take a place.

Now, let us consider the space with the symmetry group  $T_1 \otimes SO(3)$ . According to classification of all spaces with four-parametric symmetry groups given in [14], spaces with such a symmetry group form the whole class (type VIII in the mentioned classification) and they are the direct products of a temporal axes and three-dimensional spaces with the metric depending on the constant parameters  $K_{ij} = K_{ji}$  and  $D$ . The scalar representation generators of their symmetry group are

$$\Pi_\mu^{(l)} = b_{(\mu)}^\nu \partial_\nu,$$

where  $b_{(\mu)}^\nu$  are four vectors, the nonzero component of which are

$$\begin{aligned} b_{(0)0} &= -1 \quad ; \quad b_{(1)1} = \cos(y/R) \quad ; \quad b_{(2)2} = 1 \\ b_{(3)3} &= \cos(x/R) \cos(y/R) \quad ; \quad b_{(1)3} = \cos(x/R) \sin(y/R) \\ b_{(2)3} &= -\sin(x/R) \quad ; \quad b_{(3)1} = -\sin(y/R). \end{aligned}$$

It is necessary that these vectors compose an orthonormal vierbein for the group theoretical approach has meaning. It follows from here that the nonzero components of the metric tensor are

$$g_{11} = g_{22} = g_{33} = 1 \quad ; \quad g_{23} = -\sin(x/R).$$

It is easy to prove that this metric really belongs to the specified class and has nonzero parameters  $K_{11} = 2$ ,  $K_{33} = D = 1$ . Now, the Ricci rotational coefficients are

$$G_{ikl} = R^{-1} \varepsilon_{ikl}.$$

As all the relations of Sect.3, Sect.4, Sect.6 were derived only with the use of (3.5) and without using the explicit form of vierbein, then all the formulas (except the explicit solutions of the wave equations) which take place in the  $\mathbb{R}^1 \otimes S^3$  space, take place in the Petrov type VIII too. Only difference is that instead two  $SO(3)$ -subgroups remains only one, but it is enough for the construction of the invariant wave operators.

## Appendix. Some properties of $u_s^\pm(\mathbf{k})$

Let us juxtapose to each element  $g \in SO(3, 1)$  the matrix  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  of the group  $SL(2, \mathbb{C}) \sim SO(3, 1)$ . There exists its fractionally-linear transformation on the imaginary line:

$$z \mapsto z_g = \frac{az + b}{cz + d}.$$

Then we can determine the group  $SL(2, \mathbb{C})$  representation acting on the functions of  $z$  by the following way

$$f(z) \mapsto (cz + d)^{2s} f(z_{g^{-1}}).$$

As is well known (see [9]) if  $f(z)$  is the polynomial from  $z$  of the power  $2s$ , then the representation is  $\mathbf{L}^{(s,0)}$ . This means that it is possible to write down the vectors of the representation  $\mathbf{L}^{(s,0)}$  not in the form of the lines or columns, but in the form of polynomials of a power  $2s$ . Then one can obtain [15] that in such a notation  $u_s^+(\mathbf{k})$  with the arbitrary  $s$  looks like

$$u_s^+(\mathbf{k}) = \left( \frac{1 + k^3}{2} \right)^s (1 - z\bar{\rho}_{\mathbf{k}})^{2s},$$

where

$$\rho_{\mathbf{k}} = \frac{k^1 + ik^2}{1 + k^3}.$$

The quantity  $\rho_{\mathbf{k}}$  has a simple geometrical meaning. Indeed, let us make the stereographic projection of the sphere  $S^2$  where the ends of the vectors  $\mathbf{k}$  are lying, onto the plane. Here each point  $(x, y)$  of the plane is in the conformity with the complex quantity  $x + iy$ . Then, the quantity  $\rho_{\mathbf{k}}$  corresponds to the vector  $\mathbf{k}$ . This means that the fractionally-linear action of the  $SL(2, \mathbb{C})$  onto the  $\rho_{\mathbf{k}}$  is equivalent to the projective action of the  $SO(3, 1)$  onto the  $\mathbf{k}$ :  $(\rho_{\mathbf{k}})_g = \rho_{\mathbf{k}'}$ , where  $\mathbf{k}' = \mathbf{k}_g$ . Thus, it is easy to show that

$$\left( U_s(g) u_s^+(\mathbf{k}) \right) (z) = (cz + d)^{2s} u_s^+(\mathbf{k})(z_{g^{-1}}) = (\alpha_{\mathbf{k}}(g))^s u_s^+(\mathbf{k}_{g^{-1}})(z),$$

where

$$\alpha_{\mathbf{k}}(g) = \frac{1 + k^3}{1 + k_{g^{-1}}^3} (a - b\rho_{\mathbf{k}})^{-2}$$

is independent on  $s$  and  $z$ . The change of  $\Sigma(\mathbf{k}; s)$  under the transformation is determined by the  $|\alpha_{\mathbf{k}}(g)|^2$ ; to obtaining this quantity in the convenient form let us consider the case of spin  $1/2$ . Then in the usual "columnar" notation we have

$$u^+(\mathbf{k}) = \sqrt{\frac{1 + k^3}{2}} \begin{pmatrix} -\bar{\rho}_{\mathbf{k}} \\ 1 \end{pmatrix}, \quad u^-(\mathbf{k}) = \sqrt{\frac{1 + k^3}{2}} \begin{pmatrix} 1 \\ \rho_{\mathbf{k}} \end{pmatrix}.$$

It is easy to show that

$$\Sigma(\mathbf{k}; 1/2) = -\frac{1}{2} \frac{\sigma_\alpha n^\alpha}{n^4},$$

where  $\sigma^\alpha = (\boldsymbol{\sigma}, 1)$ . As

$$U(g)\sigma_\alpha n^\alpha U^\dagger(g) = \sigma_\alpha n_g^\alpha,$$

we obtain

$$|\alpha_{\mathbf{k}}(g)|^2 = \left(\frac{n_g^4}{n^4}\right)^2.$$

Putting everything together gives the equality (7.5) looked for.

## References

- [1] S.A.Pol'shin. Group Theoretical Examination of the Relativistic Wave Equations on Curved Spaces. I.Basic Principles//gr-qc/9803091
- [2] M.Carmeli, A.Malka// Found.Phys. **20**, 71 (1990)
- [3] S.A.Pol'shin. Group Theoretical Examination of the Relativistic Wave Equations on Curved Spaces. II. De Sitter and Anti-de Sitter spaces//gr-qc/9803092
- [4] Cl.Gabriel, Ph.Spindel // J.Math.Phys **38**,622 (1997)
- [5] B.Allen, Th.Jacobson // Commun. Math. Phys. **103**, 669 (1986)
- [6] H.Janssen, C.Dullemond // J.Math. Phys **27**, 2786 (1986)
- [7] E.Kawai //Nuovo Cimento A **100**, 195 (1988)
- [8] T.Inagaki, K.Ishikawa, T.Muta //Progr. Theor. Phys. **96**,847 (1996); hep-th/9512205
- [9] M.A.Naimark. Linear representations of the Lorentz group. Moscow, 1958 (in Russian)
- [10] S.Weinberg //Phys.Rev.B **133**,1318 (1964); **134**, 882 (1964)
- [11] J.Bros, U.Moschella // Rev. Math. Phys. **8**,327 (1996)
- [12] V. M. Villalba //J. Math. Phys. **31**, 1483(1990)
- [13] A.M.Perelomov. Generalized Coherent States and Their Applications. B.:Springer, 1986.
- [14] A.Z.Petrov. Einstein Spaces. L.:Pergamon,1969.
- [15] Yu.P.Stepanovsky // Teor. Mat. Fiz. **47**, 343 (1981) (in Russian)